KT trasition

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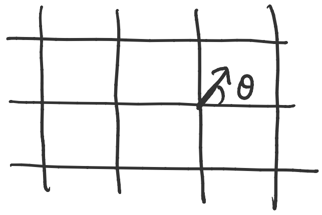
Dimension two and Kosterlitz-Thouless transition

The marginal case $d=2$ and $n=2$ for $O(n)$ model is special.
This is the 2d XY model.

$$H = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j)$$



MW says there is no long-range ordered state.

KT showed that there is still a transition of a new kind due to topological defects.

First, the possible two phases:

(1) at high temp, the spins are uncorrelated $\Rightarrow$ short range correlation.

(2) What happens at low temperature?

Low temp phase:

$$H \simeq -J\sum_{ij}\left[1 - \tfrac{1}{2}(\theta_i-\theta_j)^2\right]$$

$$\simeq -J\int d\bar{r}\left[\text{const} - (\nabla\theta)^2\right]$$

$\boxed{\text{we ignore } \theta \to \theta + 2\pi \text{ symmetry}}$

corresponding coarse-grained Landau functional

$$\mathcal{L}_{XY} = \frac{K}{2}\int_{1/\Lambda}^{L} d\bar{r}\,(\nabla\theta)^2$$

is renormalized spin coupling.

This is at low temp, where this is the relevant term.
[see Goldenfeld ch 11.2]

To analyze the nature of the low temp phase, choose a suitable local order parameter.

$$\boxed{\psi_1(\bar{r}) = e^{i\theta(\bar{r})}}$$

For the $\mathcal{L}_{XY}$, find correlation

$$\langle \psi_1(\bar{r})\psi_1^*(0)\rangle = \left\langle e^{i[\theta(r)-\theta(0)]}\right\rangle$$

$$= e^{-\frac{1}{2}\langle[\theta(\bar{r})-\theta(0)]^2\rangle}$$

because of Gaussianity of $\mathcal{L}_{XY}$.

To calculate the correlation function, go to Fourier mode.

$$\hat{\theta}(k) = \int d\bar{r}\;\theta(\bar{r})\,e^{-i\bar{k}\cdot\bar{r}}$$

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$$\Rightarrow\ e^{-\frac{R}{2}\int d\bar{r}\,(\nabla\theta)^2} \longrightarrow e^{-\frac{R}{2}\sum k^2 \hat{\theta}(k)^2}$$

This gives $\quad \langle \theta(\bar{r})\,\theta(\bar{r}')\rangle = \int \mathcal{D}[\theta(r)]\, e^{-\frac{R}{2}\int d\bar{r}(\nabla\phi)^2}\ \theta(\bar{r})\,\theta(\bar{r}')$

$$\rightarrow\ \langle \hat{\theta}(\bar{k})\hat{\theta}(\bar{k}')\rangle = \int \prod_k d\hat{\theta}(k)\cdot \hat{\theta}(k)\hat{\theta}(k')\, e^{-\frac{R}{2}\sum_k k^2\hat{\theta}(k)^2}$$

Each k-modes are decoupled. possible to do the integration for each mode.

$$= -\delta_{k,k'}\left[\frac{d}{da}\int d\theta\, e^{-a\theta^2}\right]\sqrt{\frac{a}{\pi}}\ \Bigg|_{a=\frac{Rk^2}{2}}$$

$$= \delta_{\bar{k},\bar{k}'}\cdot\frac{1}{Rk^2}$$

$$\Rightarrow\ \boxed{\langle \theta(\bar{k})\,\theta(\bar{k}')\rangle = \frac{\delta_{\bar{k},\bar{k}'}}{Rk^2}}$$

$$\Rightarrow\ \left\langle\left(\theta(\bar{r})-\theta(0)\right)^2\right\rangle = \langle\theta^2(r)\rangle + \langle\theta^2(0)\rangle - 2\langle\theta(\bar{r})\theta(0)\rangle$$

$$= \frac{2}{R}\int_{1/L}^{\Lambda}\frac{d\bar{k}}{(2\pi)^2}\ \frac{1-e^{i\bar{k}\cdot\bar{r}}}{k^2}$$

Doing the integration for $r \gg 1/\Lambda$, we get

$$\boxed{\left\langle\left(\theta(\bar{r})-\theta(0)\right)^2\right\rangle \simeq \frac{1}{R\pi}\log(r\Lambda)}$$

① The divergence with large r, confirms that there is no long-range order.

② The correlation

$$G(r) = \langle \psi_1(r)\,\psi_1^*(0)\rangle \simeq e^{-\frac{1}{2\pi R}\log(r\Lambda)}$$

$$\sim \frac{1}{(r\Lambda)^{\eta}} \quad \text{with } \eta = \frac{1}{2\pi R}$$

This means, in the low temp phase the correlations are <u>power-law</u>.

(although the exponent is not universal)

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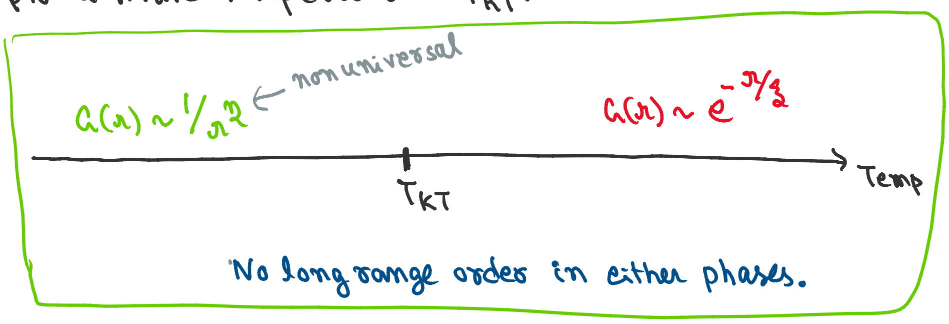
Remark: Eventhough the analysis is for low temp, the power-law phase survives upto a finite temperature T_{KT}.



Good estimate for η and R comes from RG method.

(1) at $T = T_{KT}$, $\eta = \frac{1}{4}$ universal.

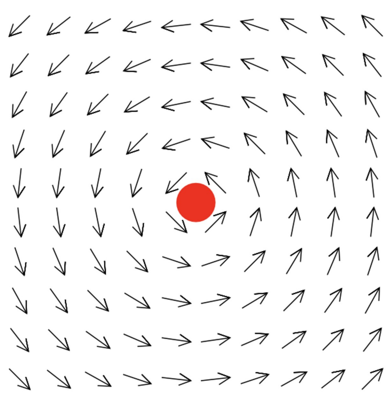
(2) for $T > T_{KT}$, $\xi \simeq e^{*/\sqrt{T - T_{KT}}}$ essential singularity.

Q. What drives this transition? A qualitative picture.

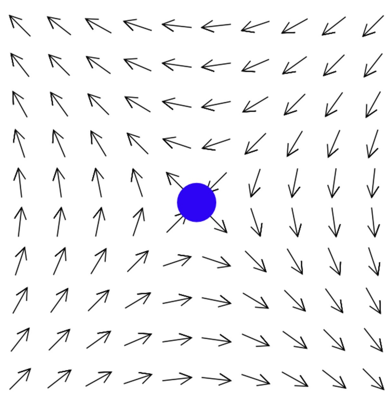
In our low temperature analysis by harmonic approximation, we ignored the $\theta \to \theta + 2\pi$ symmetry. Berezinsky and independently Kosterlitz and Thouless included most relevant features that could come from such considerations:

topological defects from singular spin configurations.

Examples of such topological defects: vortices.



$q = 1$



$q = -1$

A characteristic of a vortex is the winding number (q)

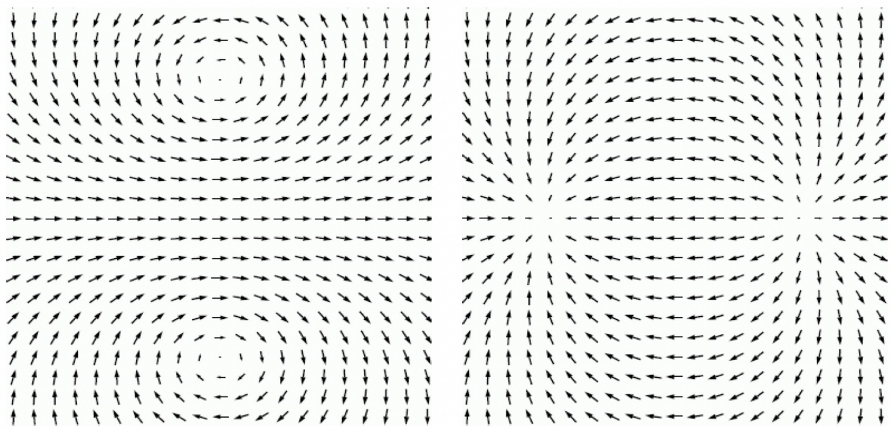
$$\boxed{\frac{1}{2\pi}\oint d\bar{l}\cdot\bar{\nabla}\theta = q}$$

along a closed path around the vortex.

[similar to Berger's vector in defect of solids]

Winding numbers are topological invariant quantities, they do not depend on the loop and they can not be changed by smooth deformation of spin orientations.

More examples of paired vortices.

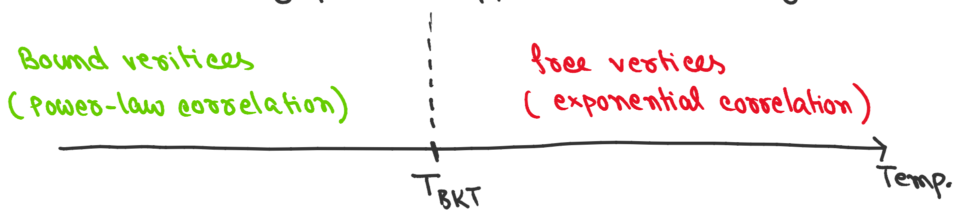


two $q=+1$ vertices two $q=+1$

The $q=0$ corresponds to a no-vortex.  Absence of vortex makes it possible to have same orientation over a long distance, therefore possible to have power-law correlation (although no long range order).

Effectively $q=0$ can be produced by a bound pair of vertices of opposit sign winding number.

In the BKT scenario the transition between power-law correlated phase and exponential decay phase happens via binding-unbinding of vortex pairs.



Based on this insight Kosterlitz-Thouless gave a Landau-Peierls type argument.

Landau-Peierls type argument:

Energy cost for a free (unbound vortex)

$$E_v \simeq \frac{R}{2}\int d\bar{r}\,(\bar{\nabla}\theta)^2$$

$$E_v \simeq \frac{R}{2}\int d\vec{r}\,(\vec{\nabla}\theta)^2$$

$$\simeq E_c + \frac{R}{2}\int_{1/\Lambda}^{L} 2\pi r\,dr\left(\frac{1}{r}\right)^2$$

[using $\nabla\theta \sim \frac{1}{r}$ for large r]

Core energy of a vertex

$$= E_c + R\pi \log(\Lambda L)$$

Entropy for placing a vertex

$$S_v \sim K_B \log(L\Lambda)$$

[number of available coarse grain boxes is $\sim \left(\frac{L}{\ell}\right)^2 = (L\Lambda)^2$]

coarse-grain scale

Then the free energy for a single unbound vortex

$$F_v = E_v - TS_v \simeq (\pi R - 2K_BT)\log(L\Lambda) + E_c$$

[Remember our excercise for Ising with $1/r^2$ interactions, and how it was the marginal case between existance of a phase transition or not]

We see that for

(1) for $\pi R < 2K_BT$, F_v decreases. Therefore unbound vertices are thermodynamically stable.

(2) for $\pi R > 2K_BT$, F_v increases. Therefore a free vertex is unstable.

A paired vertex at large distances has smaller $\nabla\theta$, therefore costs less energy. Then at

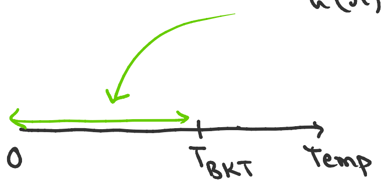
$$\boxed{T_{BKT} \simeq \frac{\pi R}{2K_B}}$$

vortices would form bound pairs.

Remark: Using small fluctuations at low temperatures we had predicted that in the low temperature phase, correlations are power-law

$$G(r) \sim \frac{1}{r^{\eta}} \quad \text{with} \quad \eta(T) = \frac{K_BT}{2\pi R}$$

In earlier derivation we had kept $K_BT = 1$.



The exponent η varies with temperatur and it is non-universal inside the phase.

If we use the result $T_{BKT} = \frac{\pi R}{2K_B}$, we get

$$\boxed{\eta(T_{BKT}) = \frac{1}{4}}$$

This value is universal and a characteristic exponent of BKT transition.

Remark: This Landau-Peierls type argument ignores interactions between vortices. The detailed analysis of Kosterlitz and Thouless considers an electrostatic-type

Remarks: This naive Peierls type argument ignores interactions between vortices.

The detailed analysis of Kosterlitz and Thouless considers an electrostatic-type effective interaction between vortices (2d coulomb gas)

winding numbers

$$\mathcal{L} = \frac{R}{2}\int d\vec{r}\,(\nabla\theta)^2 - \pi R \sum_{ij} q_i q_j \log(r_{ij}\Lambda) + \tilde{E}_c$$

is obtained by an RG calculation.

Same as what we got for energy calculation

[For details, see chapter 8 of Kardar, volume 2]

A summary sheet of the BKT transition:

$G(r) \sim \frac{1}{r^{\eta}}$

$\eta \simeq \frac{1}{4}\left[1 - * \sqrt{T_c - T}\right]$

$G(r) \sim e^{-r/\xi}$ $\quad \xi \sim e^{\frac{*}{\sqrt{T - T_c}}}$

power-law correlation | T_{BKT} | Short-range correlation

Real life examples of KT transition:

(1) Superfluid transition in bidimensional quantum system.

(2) Two-step melting in 2d particle system

[see more in Kardar's book, vol 2, ch 8.4]

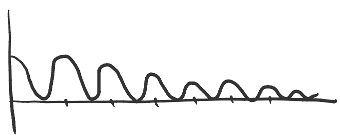
Famously known as the Berezinsky-Kosterlitz-Thouless-Halperin-Nelson-Young theory.

Summary sheet: for melting of 2d solid.

	amorphous solid	T_1	Hexatic	T_2	liquid
Positional order:	quasi-long range		short-range		short-range
orientational order:	long-range		quasi-long-range		short-range

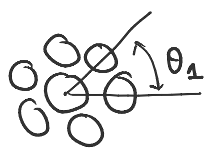
BKT transition → T

Based on position correlation: $G(r) = \langle \rho(\vec{r})\,\rho(0)\rangle$

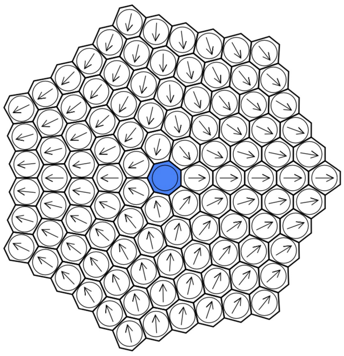


orientation correlation: $C(r) = \langle \psi_6(\vec{r})\,\psi_6^*(0)\rangle$

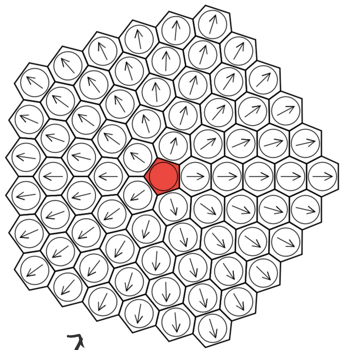
where $\psi_6 = \frac{1}{z}\sum_{k=1}^{z} e^{i\theta_k \cdot 6}$



Defects that drive this transition are called disclinations and dislocations.



This defect has 7 neighbors and $q = -1$

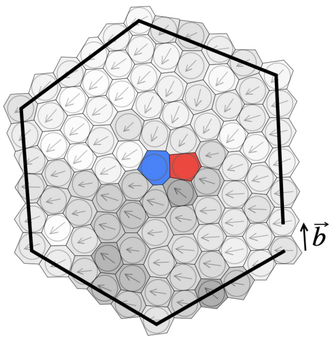


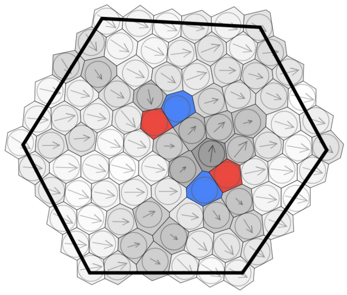
It has 5 neighbors and $q = +1$

These are disclination defects.

The arrows denote local orientation field $\psi_6(\vec{x})$. One can define winding number following similar construction as XY-model.

Individual such defects disturb long-distance orientation order, but when paired, it does not effect much at large distances.





A paired set of disclination is called a dislocation.

Individual dislocation perturbs positional order, but a paired one does not effect at large distances.

Solid	Hexatic	liquid
bound dislocation	free dislocations	free disclinations.

Ref: For a detailed discussion, see the attached chapter of a thesis.